Closing Tue, Apr. 7: 12.1, 12.2, 12.3
Closing Thu, Apr. 9: 12.4(1)(2),12.5(1)

<u>12.2 Vectors Intro</u>

Goal: Introduce vector basics.

Def'n: A **vector** is a quantity with magnitude and direction.

We depict a vector with an arrow:

a. The length is the *magnitude*.
b. The 'tail' of the arrow is called the *initial point* and the 'head' is called the *terminal point.*

If the vector is drawn with the tail at the origin and that results in the head being at the point (v_1, v_2, v_3), then we denote the vector by

$$\mathbf{v} = < v_1, v_2, v_3 >$$

Basic fact list:

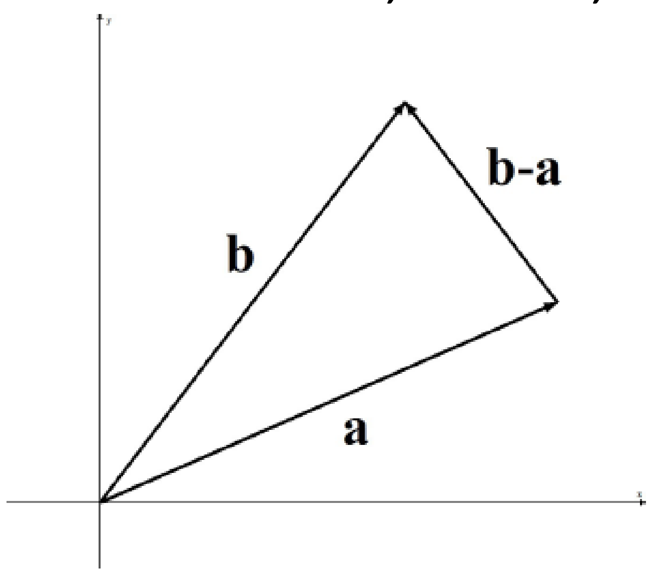
- Two vectors are equal if all components are equal.

- We denote **magnitude** by

$$|\boldsymbol{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$

- To denote the **vector from A(a_1,a_2,a_3) to B(b_1,b_2,b_3),** we write

$$\overrightarrow{AB} = \langle b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3 \rangle$$

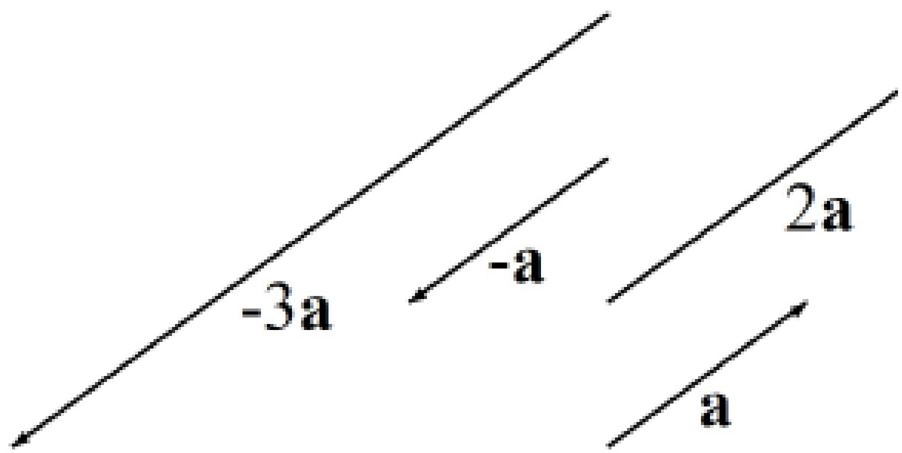


- **Scalar Multiplication**

If c is a constant, then we define

$$c\mathbf{v} = < cv_1, cv_2, cv_3 >,$$

which scales the magnitude by a factor of c.



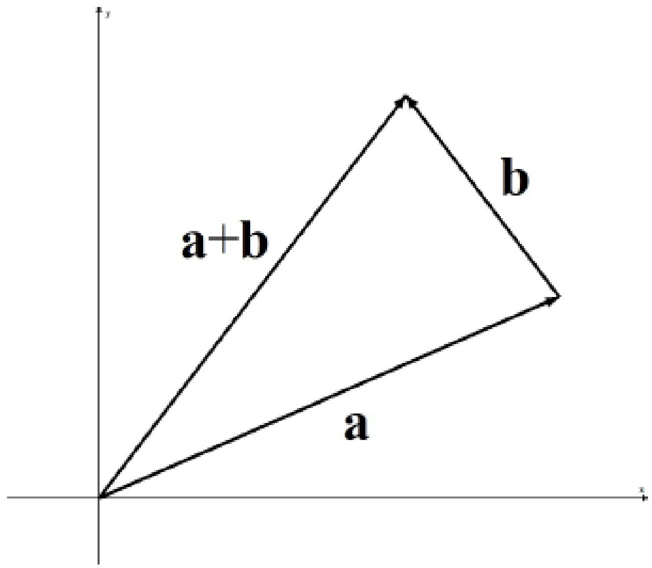
- A **unit vector** has length one.

  Note:

  $\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$ = **“**unit vector in the same direction as **v”**.

- We define the **vector sum** by

$$\mathbf{v} + \mathbf{w} = < v_1, v_2, v_3> + <w_1, w_2, w_3>$$
$$= < v_1 + w_1, v_2 + w_2, v_3 + w_3>$$



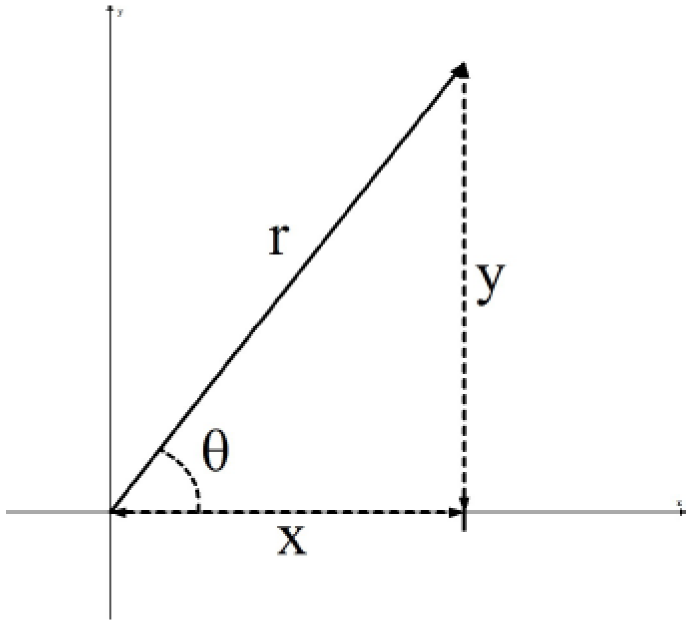
- Standard unit basis vectors:

  $\mathbf{i} = < 1, 0, 0 >$

  $\mathbf{j} = < 0, 1, 0 >$

  $\mathbf{k} = < 0, 0, 1 >$

- In 2D, you may be given the angle, θ, and length, *r*, as shown



Remember,

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$.

- In 2D, if you want a vector that is **parallel to a line with slope *m***, then the vector < 1 , m > works.

12.3 Dot Products

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and

$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Basic fact list:

- Manipulation facts (like regular multiplication):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b})$$

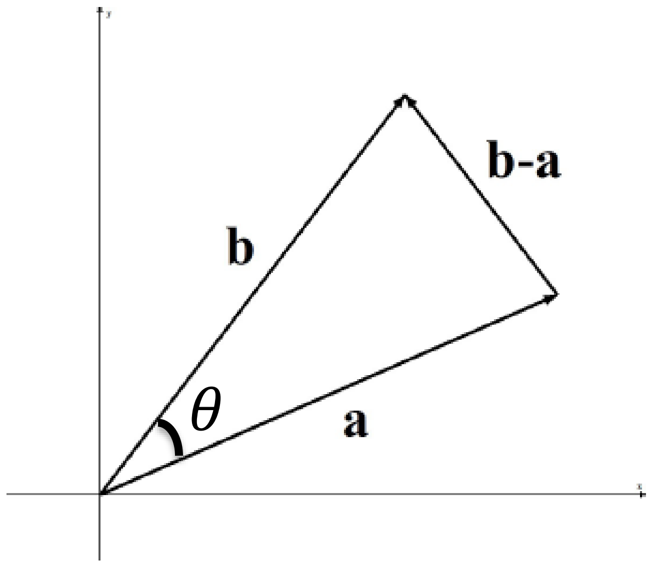
$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$$

- Helpful fact:

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$$

Most *important* dot product fact:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$



Proof (not required):

(1) By the Law of Cosines:

$$|\mathbf{b} - \mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta)$$

(2) The left-hand side expands to

$$\begin{aligned} |\mathbf{b} - \mathbf{a}|^2 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^{\mathbf{2}} \end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides of (1) yields:

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos(\theta).$$

Divide by -2 to get the result. (QED)

Most important consequence:

If **a** and **b** are orthogonal, then

$$\mathbf{a} \cdot \mathbf{b} = 0$$

Also:

If **a** and **b** are parallel, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathrm{a}||\mathrm{b}|$$

or

$$\mathbf{a} \cdot \mathbf{b} = -|\mathrm{a}||\mathrm{b}|$$

Projections: